A Euler–Poincaré framework for the multilayer Green–Nagdhi equations

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Abstract
The Green–Nagdhi equations are frequently used as a model of the wave-like behaviour of the free surface of a fluid, or the interface between two homogeneous fluids of differing densities. Here we show that their multilayer extension arises naturally from a framework based on the Euler–Poincaré theory under an ansatz of columnar motion. The framework also extends to the travelling wave solutions of the equations. We present numerical solutions of the travelling wave problem in a number of flow regimes. We find that the free surface and multilayer waves can exhibit intriguing differences compared to the results of single layer or rigid lid models.

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1. Introduction

Internal gravity waves have been observed propagating in many different locations in the world's oceans, both through direct measurement of the change in density stratification [1] and through the observed change in surface conditions, from satellite observations [2] and from recent observations from the Space Shuttle in the region of Dongsha in the South China Sea [3]. These waves play a key role in the transport of energy in the ocean and, through wave breaking, help to control mixing [4]. The observed waves are not classical one-dimensional phenomena. With wave crests extending up to 200 km perpendicular to the direction of motion, the wave properties vary with depth, exhibit curvature and interact with the underlying bathymetry and with each other. These interactions give rise to a number of phenomena including refraction, diffraction and wave front reconnection.

This paper examines one strongly nonlinear, multilayer, two-dimensional equation set for the behaviour of such waves, which is derivable from a Euler–Poincaré (EP) variational principle and shows that the equations admit travelling wave solutions which exhibit many interesting phenomena. We begin by discussing the derivation of the multilayer Green–Nagdhi (MGN) equations from a variational principle under the constraint of columnar motion. We
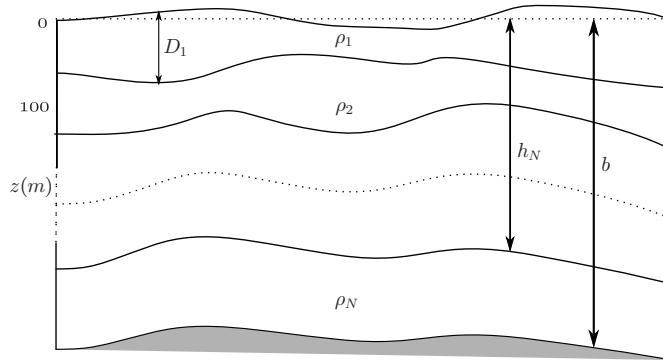
Figure 1. The multilayer fluid system.

then show that the dynamics contain a strong, fast barotropic mode and relate this to the rigid lid models of other investigators. Finally we present numerical solutions to the travelling wave problem for the system, illustrating the important behaviour of the free surface.

2. The multilayer Green–Nagdhi equations

We will begin by deriving a multilayer extension of the shallow water wave equations commonly attributed to Green and Nagdhi [5], using a variational principle under the ansatz of columnar motion. Consider a system of N homogeneous fluid layers which are acted upon by a constant gravitational acceleration, g, with the layers initially arranged in a stable stratification. Each layer thus possesses a constant density, ρ_i, for $i = 1, \ldots, N$, with $\rho_1 < \rho_2 < \cdots < \rho_N$. We have chosen by convention to number the layers downwards from the free surface (see figure 1). Let $(\mathbf{u}_i, w_i)$ denote the full three-dimensional velocity fields within the layers and h_i denote the depth of the layer interfaces with respect to the mean free-surface height at $z = 0$. We define $b = -h_{N+1}$ to be the fixed bottom topography to which the lowest layer is assumed to remain attached. Summing over the layers, the total kinetic energy of the system is

$$\int K \, \mathrm{d}\mathbf{x} := \sum_{i=1}^{N} \iint \frac{\rho_i}{2} \int_{h_{i+1}}^{h_i} |\mathbf{u}_i|^2 + w_i^2 \, \mathrm{d}z \, \mathrm{d}\mathbf{x},$$

while the total gravitational potential energy, relative to a background state with vanishing density, is

$$\int V \, \mathrm{d}\mathbf{x} := \int \sum_{i=1}^{N} \rho_i \int_{h_{i+1}}^{h_i} z \, \mathrm{d}z \, \mathrm{d}\mathbf{x} = \int \sum_{i=1}^{N} \frac{\rho_i}{2} \left(h_i^2 - h_{i+1}^2 \right) \mathrm{d}\mathbf{x}.$$

We also assume two auxiliary equations within each layer, namely a three-dimensional incompressibility condition,

$$\nabla \cdot \mathbf{u}_i + \frac{\partial w_i}{\partial z} = 0, \tag{1}$$

and a transport equation for the layer thicknesses, $D_i = h_i - h_{i+1}$, based upon the conservation of fluid within each layer:

$$\frac{\partial D_i}{\partial t} + \nabla \cdot D_i \mathbf{u}_i = 0. \tag{2}$$

We now make the ansatz of columnar motion with respect to the horizontal components of velocity in each of the layers, so that

$$\frac{\partial \mathbf{u}_i}{\partial z} = 0 \qquad \text{for} \quad i = 1, \ldots, N. \tag{3}$$

Together (1) and (3) enforce a linear dependence on z for the vertical component of velocity, w_i, so that using the bottom boundary condition,

$$\mathbf{u}_N \cdot \nabla \mathbf{b} + w_N = 0,$$

and integrating upwards, we can find the vertical velocity within each layer in terms of the horizontal velocity divergences within that layer and those below it:

$$w_i = -z \nabla \cdot \mathbf{u}_i - \nabla \cdot b\mathbf{u}_i - \sum_{j=i+1}^{N} \nabla \cdot D_j (\mathbf{u}_j - \mathbf{u}_i). \tag{4}$$

This gives a velocity jump across the interfaces of

$$w_i(h_{i+1}) - w_{i+1}(h_{i+1}) = (\mathbf{u}_{i+1} - \mathbf{u}_i) \cdot \nabla h_{i+1},$$

which is necessary to avoid the layers separating. Substituting (4) into the definition of kinetic energy and integrating in the vertical give

$$\int K \, d\mathbf{x} = \int \sum_{i=1}^{N} \frac{\rho_i}{2} \left(D_i |\mathbf{u}_i|^2 + D_i \left[\frac{D_i^2}{3} (\nabla \cdot \mathbf{u}_i)^2 + D_i \nabla \cdot \mathbf{u}_i W_i + W_i^2 \right] \right) d\mathbf{x},$$

where we have defined quantities

$$W_i := w_i(h_{i+1}) = \mathbf{u}_i \cdot \nabla \left(-b + \sum_{j=i+1}^{N} D_j \right) + \sum_{j=i+1}^{N} \nabla \cdot D_j \mathbf{u}_j,$$

which are equal to the vertical velocity at the bottom of each layer.

We now use Hamilton's principle in the form

$$\delta S := \delta \iint \ell \, d\mathbf{x} \, dt = 0,$$

applied for the reduced Lagrangian defined by

$$\ell := K - V,$$

coupled to (2). The set of physically admissible horizontal velocities in each layer (a collection of vector fields over the horizontal domain, tangential on the boundary) form a Lie algebra under what is termed the ideal fluid bracket operator where functions F, G, of a vector field $\mathbf{v}_i$, are given by

$$\{F, G\}(\mathbf{v}_i) := \int \mathbf{v}_i \cdot \left(\frac{\delta G}{\delta \mathbf{v}_i} \cdot \nabla \frac{\delta F}{\delta \mathbf{v}_i} - \frac{\delta F}{\delta \mathbf{v}_i} \cdot \nabla \frac{\delta G}{\delta \mathbf{v}_i} \right) d\mathbf{x}, \tag{5}$$

for variations defined by

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} [F(\mathbf{v} + \epsilon \delta \mathbf{v}) - F(\mathbf{v})] = \int \delta \mathbf{v} \cdot \frac{\delta F}{\delta \mathbf{v}} \, d\mathbf{x},$$

while the layer thicknesses form a set of field densities which are Lie-transported (i.e. advected) by the Eulerian velocity flow. This places the system within the formalism of Euler–Poincaré theory on semi-direct products, which extends the results of Hamiltonian mechanics into this more generalized algebraic structure.

Similar formulations have previously been derived for several alternative models for fluid flow, starting with the Euler equations, as originally considered by Poincaré and including in particular the Camassa–Holm [6] and KdV [7] shallow water equations, models which, like the single layer Green–Nagdhi equation, have been used successfully to model the behaviour of nonlinearly dispersive water waves. The Euler–Poincaré theory has connections with many other topics in geometric mechanics, notably Lagrangian reduction [8], which has been applied successfully in several areas of compressible and incompressible fluid flow and has implications for numerical methods, through a more rigorous understanding of the flow of conserved quantities in the system as geodesic motion on a suitable manifold.

Calculating the requisite adjoint actions and their duals to generate the Euler–Poincaré equations [9] in this framework (analogous to the Euler–Lagrange equations for a finite dimensional Hamiltonian system) gives the MGN equations of motion,

$$\frac{\partial \mathbf{m}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{m}_i + \mathbf{m}_i \cdot \nabla \mathbf{u}_i^T + \mathbf{m}_i \nabla \cdot \mathbf{u}_i = D_i \nabla \frac{\delta \ell}{\delta D_i}, \tag{6}$$

in terms of the layer momenta, $\mathbf{m}_i := \frac{\delta \ell}{\delta \mathbf{u}_i}$, dual to the layer velocities. The MGN equation is forced by a pressure-like term, $\frac{\delta \ell}{\delta D_i}$, containing the effects of gravity and nonhydrostatic terms, which appears out of viewing the relation between kinetic and potential energy in the system as a semi-direct product over the combined space of layer velocity and thickness fields. Explicitly, these terms are given by

$$\mathbf{m}_i = \rho_i \left[D_i \mathbf{u}_i - \nabla \left(\frac{D_i^3}{3} \nabla \cdot \mathbf{u}_i + \frac{W_i}{2} \right) - D_i \left(\frac{D_i^2}{2} \nabla \cdot \mathbf{u}_i + W_i \right) \nabla h_{i+1} \right]$$
$$- D_i \nabla \sum_{j=1}^{i-1} \rho_j D_j \left(\frac{D_i}{2} \nabla \cdot \mathbf{u}_j + W_j \right),$$

$$\frac{\delta \ell}{\delta D_i} = \frac{|\mathbf{u}_i|^2}{2} - \rho_i g h_1 - \sum_{j=1}^{i-1} (\rho_j - \rho_i) D_j$$
$$+ \sum_{j=1}^{i-1} \rho_j \left[\mathbf{u}_i \cdot \nabla \left(\frac{D_j}{2} \nabla \cdot \mathbf{u}_j + W_j \right) + \nabla \cdot \left(\frac{D_j}{2} \nabla \cdot \mathbf{u}_j + W_j \right) (\mathbf{u}_j - \mathbf{u}_i) \right].$$

Although complex, many of the terms in the equation are in balance. This is perhaps most clearly seen by rewriting (6) in the form

$$\frac{\partial}{\partial t} \left(\frac{\mathbf{m}_i}{D_i} \right) + \mathbf{u}_i \cdot \nabla \left(\frac{\mathbf{m}_i}{D_i} \right) + \left(\frac{\mathbf{m}_i}{D_i} \right) \cdot \nabla \mathbf{u}_i^T = \nabla \frac{\delta \ell}{\delta D_i}.$$

Passing the material derivative part way through the definition of the layer momenta gives

$$\frac{\mathrm{d}_i \mathbf{u}_i}{\mathrm{d}t} = \underbrace{-g \nabla \left(h_1 + \sum_{j=1}^{i-1} \frac{\rho_j - \rho_i}{\rho_i} D_j \right)}_{P_h} + \frac{1}{D_1} \nabla D_i^2 \frac{\mathrm{d}_i}{\mathrm{d}t} \left(\frac{D_i \nabla \cdot \mathbf{u}_i}{3} + \frac{W_i}{2} \right)$$
$$+ \frac{\mathrm{d}_i}{\mathrm{d}t} \left(\frac{D_i \nabla \cdot \mathbf{u}_i}{2} + W_i \right) \nabla h_{i+1} + \nabla \sum_{j=1}^{i-1} \frac{\rho_j}{\rho_i} D_j \frac{\mathrm{d}_j}{\mathrm{d}t} \left(\frac{D_j \nabla \cdot \mathbf{u}_j}{2} + W_j \right),$$

identical to the equation set given for the same fluid system in the two-layer case by Choi and Camassa [10] (henceforth the CC equation) by an asymptotic expansion method and by Liska

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and Wendroff [11] by directly substituting the definition of w_i in the Euler equations. It also agrees with the one-dimensional multilayer equation of Choi [12]. The quantity labelled P_h is a hydrostatic pressure, representing forcing from both deviations to the free-surface height and from variations in the thicknesses of the above layers. Calculating the full commutator $\left[d_i/dt, \mathcal{L}_{ij}\mathbf{u}_j \right]$ for the symmetric second-order elliptic operator defined by $\mathbf{m}_i = \mathcal{L}(D,b)_{ij}\mathbf{u}_j$, the equations may also be written entirely in terms of gradients of the horizontal velocity and layer thicknesses:

$$\mathcal{L}(D,b)\frac{d_i\mathbf{u}_i}{dt} = -\rho_i g \nabla \left(h_1 + \sum_{j=1}^{i-1} \frac{\rho_j - \rho_i}{\rho_i} D_j \right) - \frac{\rho_i}{D_i} \nabla \left(D_i^2 \left[\frac{R_i}{3} + \frac{S_i}{2} \right] \right)$$
$$+ \left(\frac{R_i}{2} + S_i \right) \nabla h_{i+1} + \nabla \sum_{j=1}^{i-1} \frac{\rho_j}{\rho_i} D_j \left(\frac{R_j}{2} + S_j \right), \tag{7}$$

where

$$R_i := D_i \left[(\nabla \cdot \mathbf{u}_i)^2 + \mathrm{tr}\left(\nabla \mathbf{u}_i^T \cdot \nabla \mathbf{u} \right) \right],$$

$$S_i := - \left[(\mathbf{u}_i \cdot \nabla \nabla h_{i+1}) \cdot \mathbf{u}_i + \sum_{j=i+1}^{N} \nabla \cdot (\nabla \cdot (D_j \mathbf{u}_j \mathbf{u}_j)) \right].$$

In this form we see the operator $\mathcal{L}$ acting as a smoother on the hydrostatic pressure, while there is an induced remainder term which grows with the nonlinearity and further couples the layers.

A final rearrangement of (6) gives an equation similar to the vorticity form of the Euler equations:

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{m}_1}{D_1} \right) + \nabla \left(\frac{\mathbf{u}_i \cdot \mathbf{m}_i}{D_i} \right) + \left[\nabla \times \left(\frac{\mathbf{m}_i}{D_i} \right) \right] \times \mathbf{u}_i = \nabla \frac{\delta \ell}{\delta D_i}.$$

Taking curls shows that the equations materially conserve a quantity,

$$q_i := \frac{1}{D_i} \nabla \times \left(\frac{\mathbf{m}_i}{D_i} \right),$$

identified with potential vorticity in each layer. These conservation laws may also be derived from the existence of individual Kelvin circulation theorems in each layer, namely

$$\frac{d}{dt} \oint_{c(\mathbf{u}_i)} \frac{\mathbf{m}_i}{D_i} \cdot d\mathbf{x} = 0,$$

where the integral is over the closed loop, c, assumed to move at the layer velocity, $\mathbf{u}_i$. In turn, it can be shown that the existence of these circulation theorems and their associated conservation laws follows in turn from the invariance of the EP formulation to fluid parcel relabelling in the configuration space, provided that it preserves Eulerian quantities. For fuller details of the Kelvin–Noether theorem for EP equations in the context of semi-direct product Lie algebras, see [9]. The integral over the entire layer volume of any function of q_i represents a conserved quantity of the MGN equations. These are the Casimirs of the Lie–Poisson Hamiltonian operator and hence of the Lie–Poisson bracket (5). That is they are functionals, C, which satisfy $\{C, H\} = 0$ for any Hamiltonian, H.

3. The barotropic and baroclinic modes

Linearizing the two-layer MGN equations around a state of rest, with standing fluid heights d_1, d_2, gives the system

$$\frac{\partial}{\partial t}\left[\mathbf{u}_1 - \frac{1}{3}d_1^2 \nabla\nabla\cdot\mathbf{u}_1 - \frac{1}{2}d_1 d_2 \nabla\nabla\cdot\mathbf{u}_2\right] = -g\nabla(D_1 + D_2),$$

$$\frac{\partial}{\partial t}\left[\mathbf{u}_2 - \frac{1}{3}d_2^2 \nabla\nabla\cdot\mathbf{u}_2 - \frac{\rho_1}{\rho_2}d_1\left(\frac{1}{2}d_1\nabla\nabla\cdot\mathbf{u}_1 + d_2\nabla\nabla\cdot\mathbf{u}_2\right)\right] = -g\nabla\left(D_2 + \frac{\rho_1}{\rho_2}D_1\right),$$

$$\frac{\partial D_1}{\partial t} + d_1\nabla\cdot\mathbf{u}_1 = 0,$$

$$\frac{\partial D_2}{\partial t} + d_2\nabla\cdot\mathbf{u}_2 = 0.$$

Taking divergences of the momentum evolution equations then gives the linearized wave equations

$$\frac{\partial^2}{\partial t^2}\left(D_1 - \frac{d_1^2}{3}\Delta D_1 - \frac{d_1 d_2}{2}\Delta D_2\right) = gd_1\Delta(D_1 + D_2),$$

$$\frac{\partial^2}{\partial t^2}\left[D_2 - \frac{d_2^2}{3}\Delta D_2 - \frac{\rho_1 d_1}{\rho_2}\Delta\left(\frac{d_1}{2}D_1 + d_2 D_2\right)\right] = gd_2\Delta\left(D_2 + \frac{\rho_1}{\rho_2}D_1\right),$$

with $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$, the standard two-dimensional Laplacian operator. Specializing to one-dimensional plane wave solutions, $D_i = c_i \exp[i(\omega t - kx)]$, we obtain a linear dispersion relation

$$\left(1 + \frac{1}{3}d_1^2 k^2 + \frac{1}{3}d_2^2 k^2 + \frac{\rho_1}{3\rho_2}d_1 d_2 k^2 + \frac{1}{9}d_1^2 d_2^2 k^4 + \frac{\rho_1}{12\rho_2}d_1^3 d_2 k^4\right)\omega^4$$

$$- gk^2\left((d_1 + d_2)\left[1 + \frac{1}{3}d_1 d_2 k^2\right] - \frac{\rho_1}{2\rho_2}d_1(d_1 - d_2)^2 k^2\right)\omega^2 + gg' d_1 d_2 k^4 = 0,$$

with $g' := g(\rho_2 - \rho_1)/\rho_2$ as a reduced gravity term. Plotting solutions shows both a fast barotropic mode, ω_f, and a slow baroclinic one, ω_s, as shown in figure 2. Under the limits $g' d_1 d_2 \ll g(d_1 + d_2)^2$, $d_i k \ll 1$, we re-obtain the shallow water modes:

$$\omega_f = \sqrt{g(d_1 + d_2)}k, \qquad \omega_s = \sqrt{g'\frac{d_1 d_2}{d_1 + d_2}}k.$$

Looking for one-dimensional solutions to the linearized system with $u_2 = \lambda u_1$ and $D_2 = (\mu - 1)D_1$, we obtain equations

$$\frac{\partial}{\partial t}\left(1 - \left[\frac{1}{3}d_1^2 + \frac{\lambda}{2}d_1 d_2\right]\frac{\partial^2}{\partial x^2}\right)u_1 = -g\mu\frac{\partial D_1}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\lambda - \left[\frac{\lambda}{3}d_2^2 + \lambda\frac{\rho_1}{\rho_2}d_1 d_2 + \frac{1}{2}\frac{\rho_1}{\rho_2}d_1^2\right]\frac{\partial^2}{\partial x^2}\right)u_1 = (-g\mu + g')\frac{\partial D_1}{\partial x}$$

$$\frac{\partial D_1}{\partial t} + d_1\frac{\partial u_1}{\partial x} = 0$$

$$(\mu - 1)\frac{\partial D_1}{\partial t} + \lambda d_2\frac{\partial u_1}{\partial x} = 0.$$
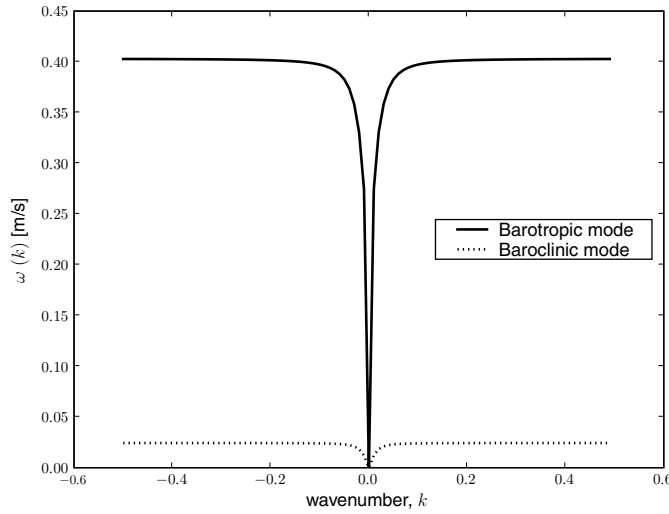
Figure 2. The linear dispersion relation for the MGN equations. In this case $d_1 = 100$ m, $\rho_1 = 1023$ kg m^{-3}, $d_2 = 200$ m, $\rho_2 = 1026$ kg m^{-3}. Note the wide separation of scales between the two modes.

The final two equations give an obvious relation

$$(\mu - 1) = \frac{d_2}{d_1}\lambda,$$

while the first two equations can be rearranged to give $\frac{\partial^2 u}{\partial x^2}$ in terms of quantities other than u:

$$\left[\frac{\lambda}{3}\left(d_1^2 - d_2^2\right) + \lambda\left(\frac{\lambda}{2} - \frac{\rho_1}{\rho_2}\right)d_1 d_2 \; - \; \frac{1}{2}\frac{\rho_1}{\rho_2}d_1^2\right]\frac{\partial}{\partial t}\frac{\partial^2 u_1}{\partial x^2} = (g\mu(\lambda - 1) + g')\frac{\partial D_1}{\partial x}.$$

On re-substituting this gives a cubic equation for λ. For g' small, one approximate solution is $\mu = 1 + d_2/d_1$, $\lambda = 1$; another consistent solution has $\mu \approx 0$, $\lambda \approx -d_1/d_2$. We identify the first solution with the barotropic mode and the second with the baroclinic. This shows that the linear dynamics of the barotropic and baroclinic modes are governed by

$$\frac{\partial u_{\mathrm{bt}}}{\partial t} = -g\frac{\partial D_{\mathrm{bt}}}{\partial x}, \qquad \frac{\partial D_{\mathrm{bt}}}{\partial t} = (d_1 + d_2)\frac{\partial u_{\mathrm{bt}}}{\partial x},$$

and

$$\frac{\partial L u_{\mathrm{bc}}}{\partial t} = -g'\frac{\partial D_{\mathrm{bc}}}{\partial x}, \qquad \frac{\partial D_{\mathrm{bc}}}{\partial t} + d_1\frac{\partial u_{\mathrm{bc}}}{\partial x} = 0,$$

respectively, where

$$Lu = \left(d_1/d_2 - \left[d_1 d_2/3 + \frac{\rho_1}{\rho_2}d_1 d_2 - \frac{1}{2}\frac{\rho_1}{\rho_2}d_1^2\right]\frac{\partial^2}{\partial x^2}\right)u,$$

the linearization of the MGN operator. This shows that to leading order, the barotropic mode dynamics are precisely those of the one-layer shallow water equations. Deviations in the free-surface height are directly balanced by transient changes in the barotropic velocity. The baroclinic mode meanwhile shows the smoothing of the reduced gravity pressure, as represented in (7).

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3.1. The rigid lid assumption

As we have shown the free-surface MGN equations exhibit a strong barotropic mode, which acts on a much faster scale than the baroclinic modes, which are of primary interest for studies of internal waves. This has important implications for numerical calculation, since for time stepping methods which are conditionally stable it will be the fast barotropic mode which sets the value of the constraint. For a kilometre deep ocean basin at hundred meter resolution $\Delta x/\Delta t \gg c = \sqrt{gH}$ requires time steps below 1 s, much too small for most practical purposes. One method frequently used to modify the dynamics and remove this difficulty is to impose a rigid top boundary at $z = 0$. Under the EP framework, this can be easily enforced using an additional constraint term in the reduced Lagrangian:

$$\int \phi \left[\sum_{i=1}^{N} D_j - b \right] \mathrm{d}\mathbf{x},$$

with $\phi[x, y]$, a Lagrange multiplier, being determined by the constraint that the term in brackets vanishes everywhere. Under application of the EP methodology an additional pressure term equal to the multiplier ϕ, thus constant in all layers, appears in the term due to the thickness densities, $\delta\ell/\delta D_i$. The value of ϕ can be obtained by applying the rigid lid condition to the derived equations and solving the resultant elliptic system. The net effect of these additional terms is to modify the dispersion relation for the system, reducing its dimension so that the original fast mode is removed. The modified dispersion relation is found to be

$$\left[1 + \frac{1}{d_1 + d_2} \left(d_1^2 d_2 \left[\frac{\rho_1}{\rho_2} + \frac{1}{3} \right] - \frac{d_1 d_2^2}{6} - \frac{\rho_1 d_1^3}{2\rho_2} \right) k^2 \right] \omega^2 - g' \frac{d_1 d_2}{d_1 + d_2} = 0,$$

which possesses only the baroclinic roots, as stated. This means that the timestep for numerical methods is then controlled by the large baroclinic mode representing most of the energy of internal solitary waves.

We observe that under the rigid lid assumption, the no-normal flow condition requires that the vertical velocity vanishes on the top surface so that

$$w(z = 0) := D_1 \nabla \cdot \mathbf{u}_1 + W_1 = 0.$$

If this substitution is made in the reduced Lagrangian before the variations are taken, then the resulting equations in the two-layer, one-dimensional, case agree exactly with the equations from the multiplier method, since in one dimension conservation of the MGN PV is automatic. Both methods produce a system identical to the CC equation for flow under a rigid lid with varying topographies.

4. Travelling wave solutions of the MGN equations

The one-dimensional GN equations are widely known to posses a travelling wave solution [5] of the form

$$D = d \left(1 + \left(\frac{c^2}{gd} - 1 \right) \operatorname{sech}^2 \left(\frac{\sqrt{3(c^2 - gd)}}{2cd} (x - ct) \right) \right),$$

$$u = c \left(1 - \frac{d}{D} \right), \tag{8}$$

where the quantity c is the group speed (and, since these are shallow water, waves phase speed) of a chosen wave. These waves exist for all c such that a condition for supercritical flow,

$$\frac{c^2}{gd} > 1, \tag{9}$$

is satisfied. This also defines a Froude number, $c/\sqrt{gd}$, for the system. The sech² form is also found for the KdV equation although the precise definition of velocity differs. The equation giving the GN travelling wave velocity, (8), follows directly from the transport equation for layer thickness (2). Examining the Lagrangian (now calculated against a flat background state) in the case $N = 1$,

$$\ell = \int \mathrm{d}x \, \mathrm{d}y \frac{\rho}{2} \left[D|\mathbf{u}|^2 + \frac{D^3}{6} \left(\frac{\partial u}{\partial x} \right)^2 - g(D - d)^2 \right],$$

we see that the travelling wave solution is a solution of the Hamiltonian formulation obtained by a direct substitution of the definition of u for a travelling wave, (8). The resulting Lagrangian is

$$\mathcal{L}(D, \mathrm{d}D/\mathrm{d}X) = \frac{\rho}{2} c^2 D \left(1 - \frac{d}{D} \right) - g(D - d)^2 + \frac{c^2}{6D} \left(\frac{\mathrm{d}D}{\mathrm{d}x} \right)^2,$$

which under the standard Legendre transform, $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, gives a canonical Hamiltonian in terms of $q = D$ of

$$\mathcal{H}(p, q) = \frac{\rho}{2} \left[\frac{c^2 d^2}{6q} p^2 - c^2 q \left(1 - \frac{d}{q} \right) + g(q - d)^2 \right].$$

The same equation can also be derived indirectly [13] by noting that the one-dimensional GN equation possesses a conserved layer momentum, $\int m/D \, \mathrm{d}x$, and that travelling waves are stationary functions of the quantity

$$Q = \int \mathcal{H} - c\frac{m}{D} \, \mathrm{d}x.$$

That both methods give the same final result follows from the invariance of the original Lagrangian ℓ to Galilean boosts, $(x, t) \rightarrow (x - ct, t)$, so that changing to a frame moving at the constant wave speed requires only the redefinition of the rest kinetic energy.

The Hamiltonian structure extends easily to the multilayer case, where each layer possesses an equation identical to (8). For general N, the MGN travelling wave solution is a homoclinic orbit of the Hamiltonian system given by

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \mathbf{p}^\mathbf{T} \mathbf{A}^{-1} \mathbf{p}$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \rho_i \left[g \left(\sum_{j=i}^{N} (q_j - d_j)^2 - \sum_{j=i+1}^{N} (q_j - d_j)^2 \right) - c^2 q_j \left(1 - \frac{d_j}{q_j} \right)^2 \right],$$

around the equilibrium point $(\mathbf{p}, \mathbf{q}) = (\mathbf{0}, \mathbf{d})$, where $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{d}$ are the N-dimensional vectors containing p_i, q_i and d_i and $\mathbf{A}$ is an $N \times N$ matrix with elements defined by

$$A_{ij} = \begin{cases} c^2 \left(\dfrac{\rho_i d_i^2}{3D_i} + \displaystyle\sum_{k=1}^{i-1} \dfrac{\rho_k d_k^2}{D_k} \right) & i = j \\[2em] \displaystyle\sum_{k=1}^{\min\{i,j\}} c^2 \dfrac{\rho_k d_k^2}{2D_k} & i \neq j \end{cases}.$$

Since $\mathbf{A}$ depends on $\mathbf{q}$, the system is not separable and the dynamics can be extremely complex. Figure 3 shows numerical solutions for the travelling wave problem, calculated by shooting
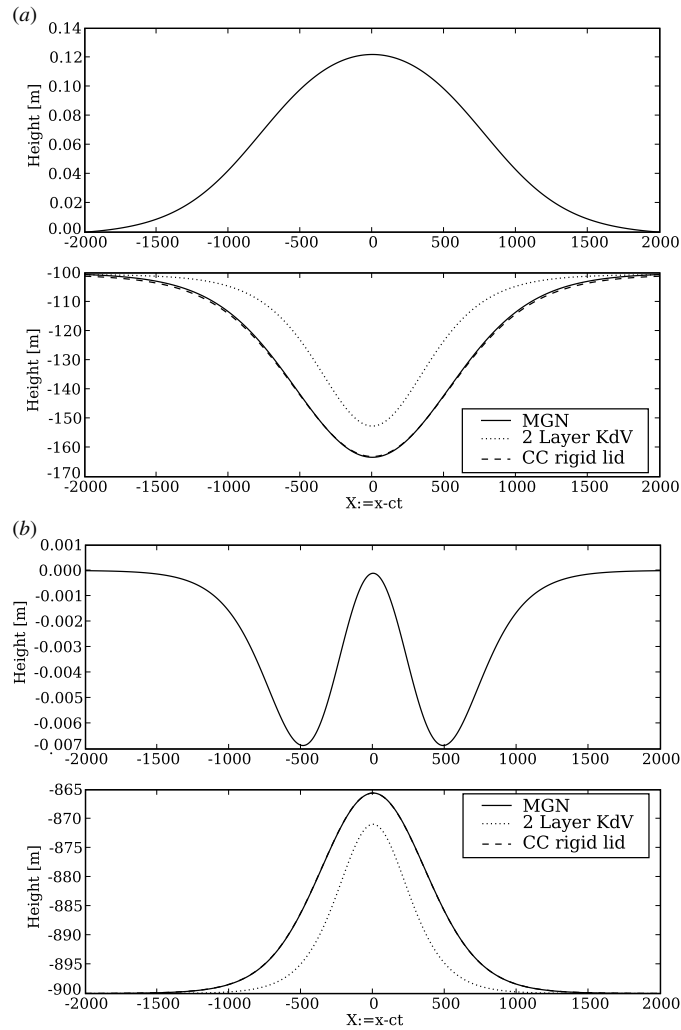
Figure 3. Numerical shooting solutions for the MGN travelling wave problem showing (*a*) a two-layer wave of depression (*b*) a two-layer wave of elevation. The CC and two-layer KdV solutions for the fluid interface are shown for reference. In both cases, the interface of the MGN wave virtually overlies the CC wave for the chosen wave speed with the KdV wave noticeably smaller in magnitude. However, the MGN equations also give a free surface deviation. The apparent dipole structure of the free surface in case (*b*) is not evident in the individual layer thicknesses which remain precisely out of phase.

along the unstable manifold from the equilibrium $(\mathbf{p}, \mathbf{q}) = (\mathbf{0}, \mathbf{d})$ at $x = -\infty$. The integration problem was found to be extremely stiff and the use of a symplectic integration method, the generalized leapfrog method, was implemented to maintain stability. For comparison, we also plot the relevant numerical solutions to the CC equations [10] and the algebraic solution to the two-layer KdV equation [14].

The condition for supercritical flow and thus existence (9) generalizes to a condition that the stable manifold of the system at $x = -\infty$ must be of dimension greater than zero. This is equivalent to the condition that $\det(\mathbf{V} - \lambda^2 \mathbf{A}) = 0$ has real solutions, where $\mathbf{V}$ is the matrix,
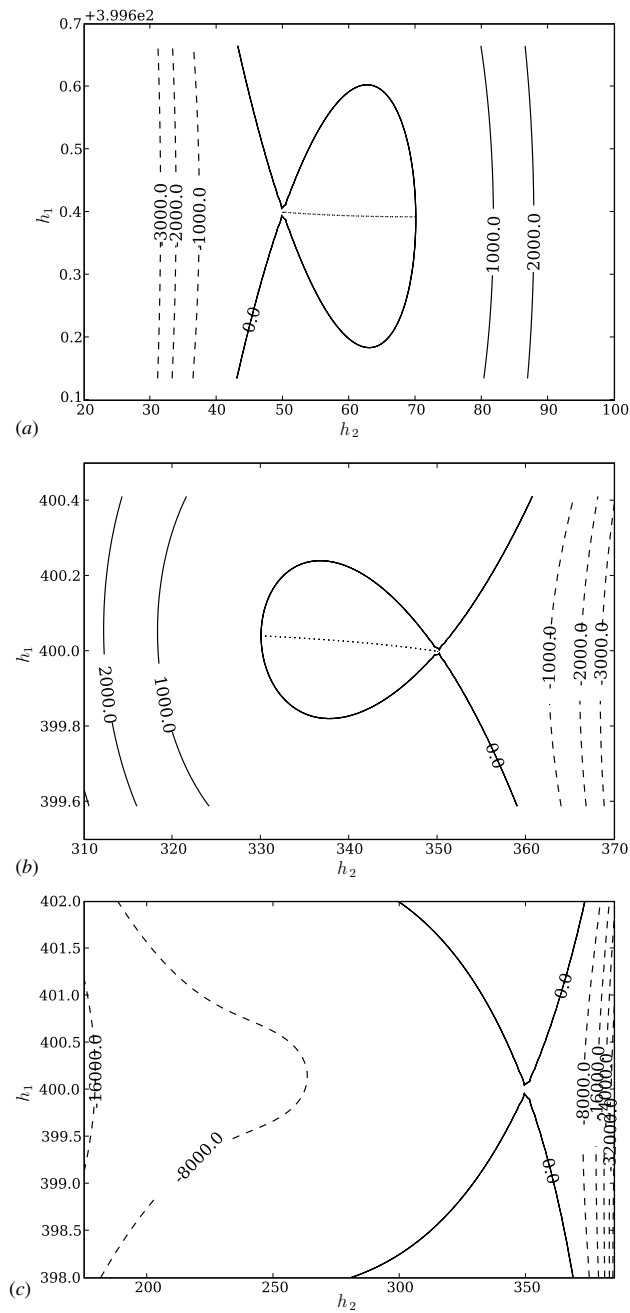
Figure 4. Contour plots of potentials which allow (*a*) a wave of elevation, (*b*) a wave of depression, (*c*) no travelling wave solutions. In the first two cases the trajectory for the travelling wave solution of the particular regime is plotted as a dotted line, contained within the zero contour. The fish-like looped zero contours disappear with increasing wave speed through a bifurcation with a second (h_1, h_2) contour joining (∞, $\pm\infty$) to ($-\infty$, $\pm\infty$) with the sign of h_2 positive for waves of elevation and negative for waves of depression.

$$V_{ij} = -\left.\frac{\partial^2 \mathcal{H}}{\partial q_i \partial q_j}\right|_{(\mathbf{p},\mathbf{q})=(\mathbf{0},\mathbf{d})} = \begin{cases} \rho_i\left(\dfrac{c^2}{d_i} - g\right) & i = j \\ -g\rho_{\min\{i,j\}} & i \neq j \end{cases}.$$

This matrix comes from the form of the Jacobian matrix of the Hamiltonian system linearized around the far field values. For $N = 2$, the critical condition for internal and external travelling waves respectively becomes

$$\frac{c^2}{g} > \frac{d_1 + d_2 \mp ((d_1 + d_2)^2 - 4(\rho_2 - \rho_1)d_1 d_2/\rho_2)^{1/2}}{2}. \tag{10}$$

This is however only a necessary condition on the wave speed and there exist stratifications for which no internal travelling wave is possible for c supercritical. This can be illustrated by considering the form of the potential

$$\mathcal{V}(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{N}\rho_i\left[g\left(\sum_{j=i}^{N}(q_j - d_j)^2 - \sum_{j=i+1}^{N}(q_j - d_j)^2\right) - c^2 q_j\left(1 - \frac{d_j}{q_j}\right)\right]$$

in the Hamiltonian in the regimes where there exist waves of elevation, waves of depression, and for cases with no travelling wave. Contour plots of the potential under these three conditions are shown in figure 4, along with the marked trajectories where a travelling wave exists. The potential is seen to undergo bifurcations with the creation or destruction of homoclinic contours around the equilibrium at $\mathbf{q} = \mathbf{d}$. The direction of the contour defines whether the wave is of elevation or depression and its destruction with increasing wave speed represents a limit on the speed (and thus amplitude) of travelling waves allowed by the system. This follows since for $N=2$, the matrix $\mathbf{A}$ is positive definite and thus $\mathbf{p}^T \mathbf{A}^{-1}\mathbf{p}$ is a strictly positive quantity. Jo and Choi [15] investigate the two-layer system in the rigid lid case and find conditions on minimum and maximum wave speed similar to those presented here. The critical flow condition is

$$\frac{c^2}{g} > \frac{(\rho_2 - \rho_1)d_1 d_2}{\rho_2(d_1 + d_2)};$$

this is the first-order term in the Taylor series expansion of (10) in the limit $(d_1 + d_2)^2 \gg 4(\rho_2 - \rho_1)d_1 d_2/\rho_2$. This suggests that the rigid lid model is a good approximation when density differences are small, or the aspect ratio d_1/d_2 differs greatly from unity, as may be expected. A similar set of calculations and analysis is in progress for the case $N = 3$, which appears to increase the dimension of possible behaviour.

5. Summary

We have introduced a set of equations derived from a variational principle under an ansatz of columnar motion. These have been shown to be identical to the multilayer Green–Nagdhi equations derived independently by other researchers by other methods. These equations are proved to contain a fast barotropic mode which is virtually unmodified by the nonlinear part of the MGN operator. This means that the equations require careful handling in numerical simulations. The travelling wave solutions are also shown to be derivable from a variational principle, and show a more complex range of behaviour and waveforms than the single layer or rigid lid case.

Acknowledgments

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